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An exact approach to the oscillator radiation process in an arbitrarily large cavity

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Abstract

Starting from a solution of the problem of a mechanical oscillator coupled to a scalar field inside a reflecting sphere of radius *R*, we study the behaviour of the system in free space as the limit of an arbitrarily large radius in the confined solution. From a mathematical point of view we show that this way of addressing the problem is not equivalent to considering the system *a priori* embedded in infinite space. In particular, the matrix elements of the transformation turning the system to the principal axis do not tend to distributions in the limit of an arbitrarily large sphere as should be the case if the two procedures were mathematically equivalent. Also, we introduce 'dressed' coordinates which allow an exact description of the oscillator radiation process. Expanding in powers of the coupling constant, we recover from our exact expressions the well known decay formulae from perturbation theory.

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1. Introduction

As is well known, the solution of coupled equations in field theory, when coupled fields (particles) are considered, is far from being an easy problem. In fact, the only available method to solve such a problem, apart from in a few special cases, is given by perturbation theory. Let us consider for instance, a charged particle described by a field $\psi(x)$ interacting with a neutral (radiation) field $\varphi(x)$ (for simplicity we drop out all spin and vector indices) through some (in general non-linear) coupling $f(g; \psi, \varphi)$, where g is some coupling constant (the charge of the particle). The perturbative solution is obtained by means of the introduction of bare, non-interacting fields $\psi_0(x)$, $\varphi_0(x)$, to which are associated bare quanta, the interaction being introduced order by order in powers of the coupling constant in the perturbative expansion. This method works remarkably well in quantum electrodynamics, weak interactions and, due to asymptotic freedom, in the high-energy domain of quantum chromodynamics. However, due to the non-vanishing of the coupling constant, the idea of a bare particle associated to the field $\psi_0(x)$ is actually an artifact of perturbation theory and is physically meaningless. The physical particle is always coupled to the radiation field; in other words, it is always 'dressed' by a cloud of quanta of the neutral field $\varphi(x)$ (photons, in the case of the electromagnetic field). In perturbation theory this dressing of the charged particle is done by the renormalization procedure, order by order in powers of the renormalized coupling constant. In practice we are limited to relatively small orders, calculations becoming very involved at higher orders.

In fact, there are situations where perturbation theory is of little use, for instance, the observation of resonant effects associated to the coupling of atoms with strong radiofrequency fields [1]. As remarked in [2], the theoretical understanding of these effects using perturbative methods requires the calculation of very high-order terms in perturbation theory, which makes the standard Feynman diagram technique practically unreliable in those cases. The trials of treating non-perturbatively such systems consisting of an atom coupled to the electromagnetic field, have led to the idea of the 'dressed atom', introduced in [3] and [4]. This approach consists of quantizing the electromagnetic field and analysing the whole system consisting of the atom coupled to the electromagnetic field. Since then, this concept has been extensively used to investigate several situations involving the interaction of atoms and electromagnetic fields, for instance, atoms embedded in a strong radiofrequency field background in [5] and [6], atoms in intense resonant laser beams in [7] or the study of photon correlations and quantum jumps. In this last situation, as shown in [8–10], the statistical properties of the random sequence of outcoming pulses can be analysed by a broadband photodetector and the dressed atom approach provides a convenient theoretical framework to perform this analysis.

Besides the idea of the dressed atom in itself, another aspect that deserves attention is the non-linear character of the problem involved in realistic situations which implies, as noted above, very hard mathematical problems to deal with. A way to circumvect these mathematical difficulties is to assume that under certain conditions the coupled-atom–electromagnetic-field system may be approximated by a system composed of a harmonic oscillator coupled *linearly* to the field through some effective coupling constant g. We consider in particular a system of this type confined to a spherical cavity of radius R. This is the case in the context of the general QED linear response theory, where the electric dipole interaction gives the leading contribution to the radiation process [11, 13]. These authors consider a radiating dipole inside a hollow spherical cavity. They calculate the dipole energy level shifts and the modified dipolar decays rates for an atom located at the centre of an empty sphere.

In this sense, in a slightly different context, a significant number of works have recently been devoted to the study of cavity QED, in particular to the theoretical investigation of higher-generation Schrödinger cat-states in high-Q cavities, as was done for instance in [14]. Linear approximations of this type have been applied in recent years in condensed matter physics for studies of Brownian motion and in quantum optics to study decoherence, by assuming a linear coupling between a cavity harmonic mode and a thermal bath of oscillators at zero temperature, as done in [15] and [16]. To investigate decoherence of higher-generation Schrödinger cat-states the cavity-field-reduced matrix for these states could be calculated either by evaluating the normal-ordering characteristic function, or by solving the evolution equation for the field-reservoir state using the normal-mode expansion, generalizing the analysis of [15] and [16].

In this paper we adopt a general physicist's point of view; we do not intend to describe the specific features of a particular physical situation. Instead we analyse a simplified linear version of the atom–field system and we try to extract the most detailed information we can from this model. We take a linear simplified model in order to try to have a clearer understanding of what we believe is one of the essential points, namely, the need for non-perturbative analytical treatments of coupled systems, which is the basic problem underlying the idea of the dressed atom. Of course, such an approach to a realistic non-linear system is an extremely hard task and here we achieve what we think is a good agreement between physical reality and mathematical reliability, with the hope that in future work our approach could be transposed to more realistic situations.

We consider a non-relativistic system composed of a harmonic oscillator coupled linearly to a scalar field in ordinary Euclidean three-dimensional space. We start from an analysis of the same system confined in a reflecting sphere of radius R, and we assume that the free-space solution to the radiating oscillator should be obtained taking a radius arbitrarily large in the R-dependent quantities. This idea of confining the system in a finite volume has been present a long time in the literature [18]. This device is introduced to make the eigenvalue problem mathematically well defined, but the limit of taking afterwards an infinite volume is non-trivial. In particular, as is stressed in the appendix of [18] the states in a continuous formulation cannot simply be considered as the infinite-volume limit of confined eigenstates. More recently, very similar ideas have been employed in radiation theory [19]. These authors introduce and develop the resolvent method for quantum mechanical systems with an infinite set of discrete levels, in view of its generalization to systems with a continuous spectrum. The limit of an arbitrarily large radius in the mathematics of the confined system is taken as a good description of the ordinary situation of the radiating oscillator in free space. We will see that this is not equivalent to the alternative continuous formulation in terms of distributions, which is the case when we consider *a priori* the system in unlimited space. The limiting procedure adopted here allows us to avoid the inherent ambiguities present in the continuous formulation. From a physical point of view we give a non-perturbative treatment to the oscillator radiation introducing some coordinates that allow us to divide the coupled system into two parts, the ‘dressed’ oscillator and the field, what makes unnecessary to work directly with the concepts of ‘bare’ oscillator, field and interaction to study the radiation process. These are the main reasons why we study a simplified linear system instead of a more realistic model: to make evident some subtleties of the mathematics involved in the limiting process of taking a cavity to be arbitrarily large, and also to exhibit a rigorous, exact solution to the radiation process by an oscillator. These aspects would be masked in the perturbative approach used to study non-linear couplings.

We start considering a harmonic oscillator $q_0(t)$ of frequency ω_0 coupled linearly to a scalar field $\phi(\boldsymbol{r}, t)$, the whole system being confined in a sphere of radius R centred at the oscillator position. The equations of motion are

$$\ddot{q}_0(t) + \omega_0^2 q_0(t) = 2\pi\sqrt{gc}\int_0^R \mathrm{d}^3\boldsymbol{r}\,\phi(\boldsymbol{r}, t)\delta(\boldsymbol{r}) \tag{1}$$

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi(\boldsymbol{r}, t) = 2\pi\sqrt{gc}\,q_0(t)\delta(\boldsymbol{r}) \tag{2}$$

which, using a basis of spherical Bessel functions defined in the domain $0 < |\boldsymbol{r}| < R$, may be written as a set of equations coupling the oscillator to the harmonic field modes

$$\ddot{q}_0(t) + \omega_0^2 q_0(t) = \eta\sum_{i=1}^{\infty}\omega_i q_i(t) \tag{3}$$

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = \eta\omega_i q_0(t). \tag{4}$$

In the above equations, g is a coupling constant, $\eta = \sqrt{2g\Delta\omega}$ and $\Delta\omega = \pi c/R$ is the interval between two neighbouring field frequencies, $\omega_{i+1} - \omega_i = \Delta\omega = \pi c/R$.

2. The transformation to the principal axis and the eigenfrequencies spectrum

2.1. Coupled harmonic oscillators

Let us consider for a moment the problem of a harmonic oscillator q_0 coupled to N other oscillators. In the limit $N \to \infty$ we recover our original situation of the coupling oscillator-field after redefinition of divergent quantities, in a manner analogous to renormalization as occurs in field theories. Systems composed of $N + 1$ coupled harmonic oscillators have been already treated in the literature. Particularly, the problem of the diagonalization of the Lee–Friedrechs Hamiltonian, which describes a two-level system interacting with a scalar field, has been studied, for instance in [17]. From the mathematical point of view the structure of our equation (20) is basically the same as that of equation (4) of [17], whose graphical solution is displayed in figure 1 of [17], what means that the eigenfrequencies spectra of both coupled systems are basically the same. This similarity reflects a formal relation between our system and previous results in the literature for the interaction of a two-level atom with a field in the rotating wave approximation (RWA), which neglects counterrotating terms in the atom–field interaction. In terms of the cutoff N the coupled equations (3) and (4) are simply rewritten taking the upper limit N instead of ∞ for the summation on the right-hand side of (3) and the system of $N + 1$ coupled oscillators $q_0, \{q_i\}$ corresponds to the Hamiltonian

$$H = \frac{1}{2}\left[p_0^2 + \omega_0^2 q_0^2 + \sum_{k=1}^{N}(p_k^2 + \omega_k^2 q_k^2 - 2\eta\omega_k q_0 q_k)\right]. \tag{5}$$

The Hamiltonian (5) can be turned to the principal axis by means of a point transformation,

$$q_\mu = t_\mu^r Q_r \qquad p_\mu = t_\mu^r P_r \tag{6}$$

performed by an orthonormal matrix $T = (t_\mu^r)$, $\mu = (0, k)$, $k = 1, 2, \ldots, N$, $r = 0, \ldots, N$. The subscripts 0 and k refer respectively to the oscillator and the harmonic modes of the field and r refers to the normal modes. The transformed Hamiltonian in the principal axis is

$$H = \frac{1}{2}\sum_{r=0}^{N}(P_r^2 + \Omega_r^2 Q_r^2) \tag{7}$$

where the Ω_r's are the normal frequencies corresponding to the possible collective oscillation modes of the coupled system.

Using the coordinate transformation $q_\mu = t_\mu^r Q_r$ in the equations of motion and explicitly making use of the normalization condition $\sum_{\mu=0}^{N}(t_\mu^r)^2 = 1$, we get

$$t_k^r = \frac{\eta\omega_k}{\omega_k^2 - \Omega_r^2} t_0^r \tag{8}$$

$$t_0^r = \left[1 + \sum_{k=1}^{N}\frac{\eta^2\omega_k^2}{(\omega_k^2 - \Omega_r^2)^2}\right]^{-\frac{1}{2}} \tag{9}$$

and

$$\omega_0^2 - \Omega_r^2 = \eta^2 \sum_{k=1}^{N}\frac{\omega_k^2}{\omega_k^2 - \Omega_r^2}. \tag{10}$$

There are $N + 1$ solutions Ω_r to (10), corresponding to the $N + 1$ normal collective oscillation modes. To gain some insight into these solutions, we take $\Omega_r = \Omega$ in (10) and transform the right-hand term. After some manipulation we obtain

$$\omega_0^2 - N\eta^2 - \Omega^2 = \eta^2 \sum_{k=1}^{N}\frac{\Omega^2}{\omega_k^2 - \Omega^2}. \tag{11}$$

It is easily seen that if $\omega_0^2 > N\eta^2$, equation (11) yields only positive solutions for Ω^2, what means that the system oscillates harmonically in all its modes. Indeed, in this case the left-hand term of (11) is positive for negative values of Ω^2. Conversely, the right-hand term is negative for those values of Ω^2. Thus there is no negative solution of that equation when $\omega_0^2 > N\eta^2$. On the other hand it can be shown that if $\omega_0^2 < N\eta^2$, equation (11) has a single negative solution Ω_-^2. In order to prove this let us define the function

$$I(\Omega^2) = (\omega_0^2 - N\eta^2) - \Omega^2 - \eta^2 \sum_{k=1}^{N} \frac{\Omega^2}{\omega_k^2 - \Omega^2}. \tag{12}$$

Accordingly (11) can be rewritten as $I(\Omega^2) = 0$. It can be seen that $I(\Omega^2) \to \infty$ as $\Omega^2 \to -\infty$ and

$$I(\Omega^2 = 0) = \omega_0^2 - N\eta^2 < 0. \tag{13}$$

Furthermore, $I(\Omega^2)$ is a monotonically decreasing function in that interval. Consequently $I(\Omega^2) = 0$ has a single negative solution when $\omega_0^2 < N\eta^2$, as we have pointed out. This means that there is an oscillation mode whose amplitude varies exponentially and that does not allow stationary configurations. We will disregard this last situation. Nevertheless, it is interesting to note that, in a different context, it is precisely this negative squared frequency solution (runaway solution) that is related to the existence of a bound state in the Lee–Friedrechs model. This solution is considered in [20] in the framework of a model to describe qualitatively the existence of bound states in particle physics. This question is also studied in the relativistic context in [21]. Thus we assume $\omega_0^2 > N\eta^2$ and define the *renormalized* oscillator frequency $\bar{\omega}$ [22],

$$\bar{\omega} = \sqrt{\omega_0^2 - N\eta^2}. \tag{14}$$

In terms of the renormalized frequency (10) becomes

$$\bar{\omega}^2 - \Omega_r^2 = \eta^2 \sum_{k=1}^{N} \frac{\Omega_r^2}{\omega_k^2 - \Omega_r^2}. \tag{15}$$

From (8), (9) and (15), a straightforward calculation shows the orthonormality relations for the transformation matrix (t_μ^r).

We get the transformation matrix elements for the oscillator-field system by taking the limit $N \to \infty$ in the above equations. Recalling the definition of η from (3) and (4), we obtain after some algebraic manipulation, from (15), (8) and (9), the matrix elements in the limit $N \to \infty$

$$t_0^r = \frac{\Omega_r}{\sqrt{\frac{R}{2\pi g c}(\Omega_r^2 - \bar{\omega}^2)^2 + \frac{1}{2}(3\Omega_r^2 - \bar{\omega})^2 + \frac{\pi g R}{2c}\Omega_r^2}} \tag{16}$$

and

$$t_k^r = \frac{\eta\omega_k}{\omega_k^2 - \Omega_r^2} t_0^r. \tag{17}$$

2.2. The eigenfrequencies spectrum

Let us now return to the coupling oscillator-field by taking the limit $N \to \infty$ in the relations of the preceding subsection. In this limit the need for the frequency renormalization in (14) becomes clear. It is exactly the analogous of a mass renormalization in field theory; the infinite

ω_0 is chosen in such a way as to make the renormalized frequency $\bar{\omega}$ finite. Recalling (15), the solutions with respect to the variable Ω of the equation

$$\bar{\omega}^2 - \Omega^2 = \frac{2\pi g c}{R} \sum_{k=1}^{\infty} \frac{\Omega^2}{\omega_k^2 - \Omega^2} \tag{18}$$

give the collective modes frequencies. We recall $\omega_k = k\frac{\pi c}{R}$, $k = 1, 2, \ldots,$ and take a positive x such that $\Omega = x\frac{\pi c}{R}$. Then using the identity

$$\sum_{k=1}^{\infty} \frac{x^2}{k^2 - x^2} = \frac{1}{2}(1 - \pi x \cot \pi x) \tag{19}$$

equation (18) may be rewritten in the form

$$\cot \pi x = \frac{c}{Rg} x + \frac{1}{\pi x}\left(1 - \frac{R\bar{\omega}^2}{\pi g c}\right). \tag{20}$$

The secant curve corresponding to the right-hand side of the above equation cuts only once each branch of the cotangent on the left-hand side. Thus we may label the solutions x_r as $x_r = r + \epsilon_r$, $0 < \epsilon_r < 1$, $r = 0, 1, 2, \ldots,$ and the collective eigenfrequencies are

$$\Omega_r = (r + \epsilon_r)\frac{\pi c}{R} \tag{21}$$

the ϵ's satisfying the equation,

$$\cot(\pi \epsilon_r) = \frac{\Omega_r^2 - \bar{\omega}^2}{\Omega_r \pi g} + \frac{c}{\Omega_r R}. \tag{22}$$

The field $\phi(\boldsymbol{r}, t)$ can be expressed in terms of the normal modes. We start from its expansion in terms of spherical Bessel functions

$$\phi(\boldsymbol{r}, t) = c \sum_{k=1}^{\infty} q_k(t)\phi_k(\boldsymbol{r}) \tag{23}$$

where

$$\phi_k(\boldsymbol{r}) = \frac{\sin \frac{\omega_k}{c}|\boldsymbol{r}|}{\boldsymbol{r}\sqrt{2\pi R}}. \tag{24}$$

Using the principal axis transformation matrix, together with the equations of motion, we obtain an expansion for the field in terms of an orthonormal basis associated to the collective normal modes

$$\phi(\boldsymbol{r}, t) = c \sum_{s=0}^{\infty} Q_s(t)\Phi_s(\boldsymbol{r}) \tag{25}$$

where the normal collective Fourier modes

$$\Phi_s(\boldsymbol{r}) = \sum_k t_k^s \frac{\sin \frac{\omega_k}{c}|\boldsymbol{r}|}{\boldsymbol{r}\sqrt{2\pi R}} \tag{26}$$

satisfy the equation

$$\left(-\frac{\Omega_s^2}{c^2} - \Delta\right)\phi_s(\boldsymbol{r}) = 2\pi\sqrt{\frac{g}{c}}\delta(\boldsymbol{r})t_0^s \tag{27}$$

which has a solution of the form

$$\phi(\boldsymbol{r}, t) = -\sqrt{\frac{g}{c}} \frac{t_0^s}{2|\boldsymbol{r}| \sin \delta_s} \sin\left(\frac{\Omega_s}{c}|\boldsymbol{r}| - \delta_s\right). \tag{28}$$

To determine the phase δ_s we expand the right-hand term of (28) and compare with the formal expansion (26). This implies the condition

$$\sin\left(\frac{\Omega_s}{c}R - \delta_s\right) = 0. \tag{29}$$

Remembering from (21) that there is $0 < \epsilon_s < 1$ such that $\Omega_s = (s + \epsilon_s)\frac{\pi}{R}$, it is easy to show from the condition in (27) that the phase $0 < \delta_s < \pi$ has the form

$$\delta_s = \epsilon_s \pi. \tag{30}$$

Comparing (24) and (26) and using the explicit form (16) of the matrix element t_0^s we obtain the expansion for the field in terms of the normal collective modes

$$\phi(\boldsymbol{r}, t) = -\frac{\sqrt{gc}}{2}\sum_s \frac{Q_s \sin\left(\frac{\Omega_s}{c}|\boldsymbol{r}| - \delta_s\right)}{|\boldsymbol{r}|\sqrt{\sin^2\delta_s + \left(\frac{\eta R}{2c}\right)^2\left(1 - \frac{\sin\delta_s\cos\delta_s}{\Omega_s R/c}\right)}}. \tag{31}$$

3. The limit $R \to \infty$: mathematical aspects

3.1. Discussion of the mathematical problem

Unless explicitly stated, in the remainder of this paper the symbol $R \to \infty$ is to be understood as the situation of a cavity of fixed, arbitrarily large radius. In order to compare the behaviour of the system in a very large cavity to what it would be in free space, let us first consider the system embedded in an *a priori* infinite Euclidean space; in this case to compute the quantities describing the system means essentially to replace by integrals the discrete sums appearing in the confined problem, taking directly $R = \infty$. An alternative procedure is to compute the quantities describing the system confined in a sphere of radius R and take the limit $R \to \infty$ afterwards. This last approach to describe the system in free space should retain in some way the 'memory' of the confined system. To be physically equivalent one should expect that the two approaches give the same results. We will see that at least from a mathematical point of view this is not exactly the case. We remark that solutions to the problem of a system composed of an oscillator coupled to a field in free space, have been known for a long time [23, 24], in the context of Brownian motion. These solutions are different from ours, in the sense that they do not consider the free-space solution as a limiting case of the solution to the system initially confined inside a box.

In the continuous formalism of free space the field normal modes Fourier components (analogous to the components ϕ_s in (26)) are

$$\phi_\Omega = h(\Omega)\int_0^\infty d\omega\,\frac{\omega}{\omega^2 - \Omega^2}\frac{\sin\frac{\omega}{c}|\boldsymbol{r}|}{|\boldsymbol{r}|} \tag{32}$$

where

$$h(\Omega) = \frac{2g\Omega}{\sqrt{(\Omega^2 - \bar{\omega}^2)^2 + \pi g^2\Omega^2}} \tag{33}$$

and where the we have taken the appropriate continuous form of (16) and (17). Splitting $\omega/(\omega^2 - \Omega^2)$ into partial fractions we get

$$\phi_\Omega = h(\Omega)\int_{-\infty}^{+\infty} d\omega\,\frac{1}{\omega - \Omega}\frac{\sin\frac{\omega}{c}|\boldsymbol{r}|}{|\boldsymbol{r}|}. \tag{34}$$

The pole at $\omega = \Omega$ prevents the existence of the integral in (34). The usual way to circumvect this difficulty is to replace the integral by one of the quantities

$$\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{1}{\omega - (\Omega \pm i\epsilon)} \frac{\sin\frac{\omega}{c}|\boldsymbol{r}|}{|\boldsymbol{r}|} \equiv \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \delta_{\pm}(\omega - \Omega) \frac{\sin\frac{\omega}{c}|\boldsymbol{r}|}{|\boldsymbol{r}|} \tag{35}$$

where

$$\delta_{\pm}(\omega - \Omega) = \frac{1}{\pi} P\left(\frac{1}{\omega - \Omega}\right) \pm \mathrm{i}\delta(\omega - \Omega) \tag{36}$$

with P standing for principal value. In our case this redefinition of the normal-mode Fourier components may be justified by the fact that both integrals in (35) are solutions of the equations of motion (1) and (2) for $\boldsymbol{r} \neq 0$, and so the solution should be a linear combination of them. The situation is different if we adopt the point of view of taking the limit $R \to \infty$ in the solution of the confined problem. In this case the Fourier component ϕ_Ω is obtained by taking the limit $R \to \infty$ in the expression for the field, equation (28), which allows us to obtain a uniquely defined expression to the normal-mode Fourier components, a phase δ_Ω corresponding to each ϕ_Ω (the limit $R \to \infty$ of δ_s in (22)) given by

$$\cot\delta_\Omega = \frac{1}{\pi g}\frac{\Omega^2 - \bar{\omega}^2}{\Omega}. \tag{37}$$

Also, comparing (35), (36) and (26) we see that the adoption of the continuous formalism is equivalent to assuming that in the limit $R \to \infty$ the elements t_i^s of the transformation matrix should be replaced by $\delta_+(\omega - \Omega)$ or by $\delta_-(\omega - \Omega)$. This procedure is, from a mathematical point of view, perfectly justified but at the price of losing uniqueness in the definition of the field components.

If we take the solution of the confined problem and we compute the matrix elements t_i^s for R arbitrarily large, we will see in subsection 3.2 that these elements do not tend to distributions in this limit. As R becomes larger the set of non-vanishing elements t_i^s concentrates for each i in a small neighbourhood of ω_i. In the limit $R \to \infty$ the whole set of the matrix elements t_i^s contains an arbitrarily large number of elements quadratically summable. For the matrix elements t_0^s we obtain a quadratically integrable expression.

In the continuous formulation the unit matrix, corresponding to the absence of coupling, has elements $E_\omega^\Omega = \delta(\omega - \Omega)$ while, if we start from the confined situation, it can be verified that in the limit $g \to 0$, $R \to \infty$, the matrix $T = (t_\mu^s)$ tends to the usual unit matrix of elements $E_{\omega,\Omega} = \delta_{\omega,\Omega}$.

The basic quantity describing the system, the transformation matrix $T = (t_\mu^s)$ has, as we will see, different properties in free space, if we use the continuous formalism or if we adopt the procedure of taking the limit $R \to \infty$ from the matrix elements in the confined problem . In the first case we must define the matrix elements t_ω^Ω linking free-field modes to normal modes, as distributions. On the other hand, adopting the second procedure we will find that the limiting matrix elements $\mathrm{Lim}_{R\to\infty}\, t_i^s$ are not distributions, but well defined finite quantities. The two procedures are not equivalent; the limit $R \to \infty$ does not commute with other operations. In other words, if we consider the system inside a sphere of radius R, the matrix elements t_i^s describing the system form a countable set of finite elements whatever the value of the radius R, no matter how large. This is not the case if we take the system *a priori* embedded in free space. The two mathematical languages are not equivalent, which is an indication of the non-trivial character of the transition from the discrete to the continuous, a fact already known from mathematical physicists (see for instance [18]). In this paper we take as physically meaningful the second procedure; we first solve the problem in the confined case (finite R) and take afterwards the limit of infinite (in the sense of arbitrarily large) radius of the cavity. In the next subsection we perform a detailed analysis of the limit $R \to \infty$ of the transformation matrix (t_μ^r).

3.2. The transformation matrix in the limit $R \to \infty$

From (16) and (17) we obtain for R arbitrarily large

$$t_0^r \to \operatorname*{Lim}_{\Delta\Omega\to 0} t_{\bar\omega}^{\Omega}\sqrt{\Delta\Omega} = \operatorname*{Lim}_{\Delta\Omega\to 0} \frac{\sqrt{2g}\Omega\sqrt{\Delta\Omega}}{\sqrt{(\Omega^2-\bar\omega^2)^2+\pi^2g^2\Omega^2}} \tag{38}$$

and

$$t_k^r = \frac{2g\omega_k\Delta\omega}{(\omega_k+\Omega_r)(\omega_k-\Omega_r)}\frac{\Omega_r}{\sqrt{(\Omega_r^2-\bar\omega^2)^2+\pi^2g^2\Omega_r^2}} \tag{39}$$

where we have used the fact that in this limit $\Delta\omega = \Delta\Omega = \frac{\pi c}{R}$. The matrix elements $t_{\bar\omega}^{\Omega}$ are quadratically integrable to one, $\int (t_{\bar\omega}^{\Omega})^2\,\mathrm{d}\Omega = 1$, as may be seen using the Cauchy theorem.

For R arbitrarily large ($\Delta\omega = \frac{\pi c}{R} \to 0$), the only non-vanishing matrix elements t_i^r are those for which $\omega_i - \Omega_r \approx \Delta\omega$. To get explicit formulae for these matrix elements in the limit $R \to \infty$, let us consider R large enough such that we may take $\Delta\omega \approx \Delta\Omega$ and consider the points of the spectrum of eigenfrequencies Ω inside and outside a neighbourhood η (defined in (3) and (4) of ω_i. We note that $R > \frac{2\pi c}{g}$ implies $\frac{\eta}{2} > \Delta\omega$, then we may consider R such that the right (left) neighbourhood $\frac{\eta}{2}$ of ω_i contains an integer number, κ, of frequencies Ω_r

$$\kappa\Delta\omega = \frac{\eta}{2} = \sqrt{\frac{g\Delta\omega}{2}}. \tag{40}$$

If R is arbitrarily large we see from (40) that $\frac{\eta}{2}$ is arbitrarily small, but κ grows at the same rate, what means firstly that the difference $\omega_i - \Omega_r$ for the Ω_r's outside the neighbourhood η of ω_i is arbitrarily larger than $\Delta\omega$, implying that the corresponding matrix elements t_i^r tend to zero (see (39)). Secondly all frequencies Ω_r inside the neighbourhood η of ω_i are arbitrarily close to ω_i, being arbitrarily large in number. Only the matrix elements t_i^r corresponding to these frequencies Ω_r inside the neighbourhood η of ω_i are different from zero. For these we make the change of labels

$$r = i - n\ \left(\omega_i - \frac{\eta}{2} < \Omega_r < \omega_i\right) \qquad r = i + n\ \left(\omega_i > \Omega_r > \omega_i + \frac{\eta}{2}\right) \tag{41}$$

$i = 1, 2, \ldots$. We get, from (39),

$$t_i^i = \frac{g\omega_i}{\sqrt{(\Omega_r^2-\bar\omega^2)^2+\pi^2g^2\omega_i^2}}\frac{1}{\epsilon_i} \tag{42}$$

and

$$t_i^{i\pm n} = \mp\frac{g\omega_i}{\sqrt{(\Omega_r^2-\bar\omega^2)^2+\pi^2g^2\omega_i^2}}\frac{1}{n\pm\epsilon_i} \tag{43}$$

where ϵ_i satisfies (22) in this case

$$\cot(\pi\epsilon_i) = \frac{\omega_i^2-\bar\omega^2}{\omega_i\pi g}. \tag{44}$$

Using the formula

$$\pi^2\mathrm{cosec}^2(\pi\epsilon_i) = \frac{1}{\epsilon_i} + \sum_{n=1}^{\infty}\left[\frac{1}{(n+\epsilon_i)^2} + \frac{1}{(n-\epsilon_i)^2}\right] \tag{45}$$

it is easy to show the normalization condition for the matrix elements (42) and (43),

$$(t_i^i)^2 + \sum_{n=1}^{\infty}(t_i^{i-n})^2 + (t_i^{i+n})^2 = 1 \tag{46}$$

and also the orthogonality relation

$$\sum_r t_i^r t_k^r = 0 \qquad (i \neq k) \tag{47}$$

in the limit $R \to \infty$.

3.3. The transformation matrix in the limit $g = 0$

From (16) we get for arbitrary R

$$\operatorname*{Lim}_{g\to 0} t_0^r = \begin{cases} 1 & \text{if } \Omega_r = \bar{\omega} \\ 0 & \text{otherwise.} \end{cases} \tag{48}$$

From (42) and (43) we see that the matrix elements t_i^r for $i \neq r$ all vanish for $g = 0$. Also, using (21) we obtain for small g

$$t_i^i \approx \frac{2g\Omega_i\omega_i}{(\Omega_i^2 - \bar{\omega}^2)(\omega_i + \Omega_i)} \frac{1}{\epsilon_i} \tag{49}$$

or, expanding ϵ_i for small g from (44)

$$t_i^i(g = 0) = 1. \tag{50}$$

We see from the above expressions that in the limit $R \to \infty$ the matrix (t_μ^r) remains an orthonormal matrix in the usual sense as for finite R. With the choice of the procedure of taking the limit $R \to \infty$ from the confined solution, the matrix elements do not tend to distributions in the free-space limit as would be the case using the continuous formalism. All non-vanishing matrix elements t_i^r are concentrated inside a neighbourhood η of ω_i; their set is a quadratically summable enumerable set. The elements (t_0^r) tend to a quadratically integrable expression.

4. The radiation process

We begin this section defining some coordinates q_0', q_i' associated with the 'dressed' mechanical oscillator and to the field. These coordinates will reveal themselves to be suitable to give an appealing non-perturbative description of the oscillator-field system. For a recent account on cavity electrodynamics the reader is referred to [13] and, for a historical reference on the perturbative treatment of the oscillator-field system, to 26. The general conditions that the dressed coordinates must satisfy, taking into account that the system is rigorously described by the collective normal coordinates modes Q_r, are the following.

- Given the linear character of our problem, the coordinates q_0', q_i' should be linear functions of the collective coordinates Q_r.
- They should allow us to construct orthogonal configurations corresponding to the separation of the system into two parts, the dressed oscillator and the field.
- The set of these configurations should contain the ground state, Γ_0.

The last of the above conditions restricts the transformation between the coordinates q'_μ, $\mu = 0, i = 1, 2, \ldots$, and the collective ones Q_r to those leaving invariant the quadratic form

$$\sum_r \Omega_r Q_r^2 = \bar{\omega}(q'_0)^2 + \sum_i \omega_i (q'_i)^2. \tag{51}$$

Our configurations will behave in a first approximation as independent states, but they will evolve as time goes on, as if transitions among them were in progress, while the basic configuration Γ_0 represents a rigorous eigenstate of the system and does not change with time. The new coordinates q'_μ describe dressed configurations of the oscillator and field quanta.

4.1. The dressed coordinates q'_μ

The eigenstates of our system are represented by the normalized eigenfunctions,

$$\phi_{n_0 n_1 n_2 \ldots}(Q, t) = \prod_s \left[N_{n_s} H_{n_s} \left(\sqrt{\frac{\Omega_s}{\hbar}} Q_s \right) \right] \Gamma_0 \mathrm{e}^{-\mathrm{i} \sum_s n_s \Omega_s t} \tag{52}$$

where H_{n_s} is the n_sth Hermite polynomial, N_{n_s} is a normalization coefficient

$$N_{n_s} = (2^{-n_s} n_s!)^{-\frac{1}{2}} \tag{53}$$

and Γ_0 is a normalized representation of the ground state

$$\Gamma_0 = \exp\left[-\sum_s \frac{\Omega_s Q_s^2}{2\hbar} - \frac{1}{4} \ln \frac{\Omega_s}{\pi\hbar} \right]. \tag{54}$$

To describe the radiation process, having as initial condition that only the mechanical oscillator, q_0 be excited, the usual procedure is to consider the interaction term in the Hamiltonian written in terms of q_0, q_i as a perturbation, which induces transitions among the eigenstates of the free Hamiltonian. In this way it is possible to approximately treat the problem, having as the initial condition that only the bare oscillator be excited. But, as is well known, this initial condition is physically not consistent due to the divergence of the bare oscillator frequency if there is interaction with the field. The traditional way to circumvent this difficulty is by the renormalization procedure, introducing perturbatively order-by-order corrections to the oscillator frequency. Here we adopt an alternative procedure. We do not make explicit use of the concepts of interacting bare oscillator and field, described by the coordinates q_0 and $\{q_i\}$; we introduce 'dressed' coordinates q'_0 and $\{q'_i\}$ for, respectively the 'dressed' oscillator and the field, defined by

$$\sqrt{\frac{\bar{\omega}_\mu}{\hbar}} q'_\mu = \sum_r t_\mu^r \sqrt{\frac{\Omega_r}{\hbar}} Q_r \tag{55}$$

valid for arbitrary R, which satisfy the condition to leave invariant the quadratic form (51) and where $\bar{\omega}_\mu = \bar{\omega}$, $\{\omega_i\}$. In terms of the bare coordinates the dressed coordinates are expressed as

$$q'_\mu = \sum_\nu \alpha_{\mu\nu} q_\nu \tag{56}$$

where

$$\alpha_{\mu\nu} = \frac{1}{\sqrt{\bar{\omega}_\mu}} \sum_r t_\mu^r t_\nu^r \sqrt{\Omega_r}. \tag{57}$$

As R becomes larger we get for the various coefficients α in (57):

(*a*) From (38)

$$\lim_{R\to\infty} \alpha_{00} = \frac{1}{\sqrt{\bar{\omega}}}\int_0^\infty \frac{2g\Omega^2\sqrt{\Omega}\,\mathrm{d}\Omega}{(\Omega^2-\bar{\omega}^2)^2+\pi^2 g^2\Omega^2} \equiv A_{00}(\bar{\omega}, g). \tag{58}$$

(*b*) To evaluate α_{0i} and α_{0i} in the limit $R \to \infty$, we remember from the discussion in subsection 3.2 that in the limit $R \to \infty$, for each i the only non-vanishing matrix elements t_i^r are those for which the corresponding eigenfrequencies Ω_r are arbitrarily near the field frequency ω_i. We obtain from (38), (42) and (43)

$$\lim_{R\to\infty} \alpha_{i0} = \lim_{\Delta\omega\to 0} \frac{1}{\sqrt{\omega_i}} \frac{(2g^2\omega_i^5\Delta\omega)^{\frac{1}{2}}}{(\omega_i^2-\bar{\omega}^2)^2+\pi^2 g^2\omega_i^2}\left(\sum_{n=1}^{\infty}\frac{2\epsilon_i}{n^2-\epsilon_i^2}-\frac{1}{\epsilon_i}\right) \tag{59}$$

and

$$\lim_{R\to\infty} \alpha_{0i} = \lim_{\Delta\omega\to 0} \frac{1}{\sqrt{\bar{\omega}}} \frac{(2g^2\omega_i^5\Delta\omega)^{\frac{1}{2}}}{(\omega_i^2-\bar{\omega}^2)^2+\pi^2 g^2\omega_i^2}\left(\sum_{n=1}^{\infty}\frac{2\epsilon_i}{n^2-\epsilon_i^2}-\frac{1}{\epsilon_i}\right). \tag{60}$$

(*c*) Since in the limit $R \to \infty$ the only non-zero matrix elements t_i^r corresponds to $\Omega_r = \omega_i$, the product $t_i^r t_k^r$ vanishes for $\omega_i \neq \omega_k$. Then we obtain from (57) and (46)

$$\lim_{R\to\infty} \alpha_{ik} = \delta_{ik}. \tag{61}$$

Thus, from (56), (61), (59), (60) and (58) we can express the dressed coordinates q'_μ in terms of the bare ones, q_μ in the limit $R \to \infty$,

$$q'_0 = A_{00}(\bar{\omega}, g)q_0 \tag{62}$$

$$q'_i = q_i. \tag{63}$$

It is interesting to compare (56) with (62) and (63). In the case of (56) for finite R, the coordinates q'_0 and $\{q'_i\}$ are all dressed, in the sense that they are all collective; both the field modes and the mechanical oscillator cannot be separated in this language. In the limit $R \to \infty$, (62) and (63) tell us that the coordinate q'_0 describes the mechanical oscillator modified by the presence of the field in an insoluable way; the mechanical oscillator is always dressed by the field. On the other hand, the dressed harmonic modes of the field, described by the coordinates q'_i, are identical to the bare field modes; in other words, the field retains in the limit $R \to \infty$ its proper identity, while the mechanical oscillator is always accompanied by a cloud of field quanta. Therefore we identify the coordinate q'_0 as the coordinate describing the mechanical oscillator dressed by its proper field, being the whole system divided into dressed oscillator and field, without recourse to the concept of interaction between them, this being absorbed in the dressing cloud of the oscillator. In the next subsections we use the dressed coordinates to describe the radiation process.

4.2. *Dressed configurations and the radiation process*

For clarity of language we use in the following the term 'photon' to indicate the field quanta, even if we are dealing with a scalar field instead of the electromagnetic field. We thus speak of emission or absorbtion of a photon by the oscillator, understood as a quantum of the scalar field. Let us define for a fixed instant the complete orthonormal set of functions,

$$\psi_{\kappa_0\kappa_1\ldots}(q') = \prod_\mu \left[N_{\kappa_\mu} H_{\kappa_\mu}\left(\sqrt{\frac{\bar{\omega}_\mu}{\hbar}}q'_\mu\right)\right]\Gamma_0 \tag{64}$$

where $q'_\mu = q'_0, q'_i, \bar{\omega}_\mu = \bar{\omega}, \omega_i$ and N_{κ_μ} and Γ_0 are as in (52). Using (55) the functions (64) can be expressed in terms of the normal coordinates Q_r. But since (52) is a complete set of orthonormal functions, the functions (64) may be written as linear combinations of the eigenfunctions of the coupled system (we take $t = 0$ for the moment)

$$\psi_{\kappa_0\kappa_1\ldots}(q') = \sum_{n_0 n_1\ldots} T^{n_0 n_1\ldots}_{\kappa_0\kappa_1\ldots}(0)\phi_{n_0 n_1 n_2\ldots}(Q, 0) \tag{65}$$

where the coefficients are given by

$$T^{n_0 n_1\ldots}_{\kappa_0\kappa_1\ldots}(0) = \int \mathrm{d}Q\, \psi_{\kappa_0\kappa_1\ldots}\phi_{n_0 n_1 n_2\ldots} \tag{66}$$

the integral extending over the whole Q-space.

We consider the particular configuration ψ in which only one dressed oscillator q'_μ is in its Nth excited state

$$\psi_{0\ldots N(\mu)0\ldots}(q') = N_N H_N\left(\sqrt{\frac{\bar{\omega}_\mu}{\hbar}}q'_\mu\right)\Gamma_0. \tag{67}$$

The coefficients (66) can be calculated in this case using (66), (64) and (55) with the help of the theorem [25]

$$\frac{1}{m!}\left[\sum_r (t^r_\mu)^2\right]^{\frac{m}{2}} H_N\left(\frac{\sum_r t^r_\mu\sqrt{\frac{\Omega_r}{\hbar}}Q_r}{\sqrt{\sum_r (t^r_\mu)^2}}\right)$$
$$= \sum_{m_0+m_1+\ldots=N} \frac{(t^0_\mu)^{m_0}(t^1_\mu)^{m_1}\ldots}{m_0!m_1!\ldots} H_{m_0}\left(\sqrt{\frac{\Omega_0}{\hbar}}Q_0\right) H_{m_1}\left(\sqrt{\frac{\Omega_1}{\hbar}}Q_1\right)\ldots. \tag{68}$$

We get

$$T^{n_0 n_1\ldots}_{0\ldots N(\mu)0\ldots} = \left(\frac{m!}{n_0!n_1!\ldots}\right)^{\frac{1}{2}}(t^0_\mu)^{n_0}(t^1_\mu)^{n_1}\ldots \tag{69}$$

where the subscripts $\mu = 0, i$ refer respectively to the dressed mechanical oscillator and the harmonic modes of the field, and the quantum numbers are submitted to the constraint $n_0 + n_1 + \cdots = N$.

In the following we study the behaviour of the system with the initial condition that only the dressed mechanical oscillator q'_0 be in the Nth excited state. We will study in detail the particular cases $N = 1$ and $N = 2$, which will be enough to have a clear understanding of our approach.

N = 1. Let us call Γ^μ_1 the configuration in which only the dressed oscillator q'_μ is in the first excited level. The initial configuration in which the dressed mechanical oscillator is in the first excited level is Γ^0_1. We have from (67), (65) (69) and (55) the following expression for the time evolution of the first-level excited dressed oscillator q'_μ

$$\Gamma^\mu_1 = \sum_\nu f^{\mu\nu}(t)\Gamma^\nu_1(0) \tag{70}$$

where the coefficients $f^{\mu\nu}(t)$ are given by

$$f^{\mu\nu}(t) = \sum_s t^s_\mu t^s_\nu \mathrm{e}^{-\mathrm{i}\Omega_s t} \tag{71}$$

That is, the initially excited dressed oscillator naturally distributes its energy among itself and all other dressed oscillators, as time increases. If the mechanical dressed oscillator is in its first excited state at $t = 0$, its decay rate may evaluated from its time evolution equation

$$\Gamma^0_1 = \sum_\nu f^{0\nu}(t)\Gamma^\nu_1(0). \tag{72}$$

In (72) the coefficients $f^{0\nu}(t)$ have a simple interpretation: remembering (62) and (63), $f^{00}(t)$ and $f^{0i}(t)$ are respectively the probability amplitudes that at time t the dressed mechanical oscillator will still be excited or have radiated a 'photon' of frequency ω_i. We see that this formalism allows a quite natural description of the radiation process as a simple exact time evolution of the system. Let us for instance evaluate the oscillator decay probability in this language. From (38) and (71) we get

$$f^{00}(t) = \int_0^\infty \frac{2g\Omega^2 e^{-i\Omega t}\,d\Omega}{(\Omega^2 - \omega^2)^2 + \pi^2 g^2 \Omega^2}. \tag{73}$$

The above integral can be evaluated by the Cauchy theorem. For large t ($t \gg \frac{1}{\bar{\omega}}$), but arbitrary coupling g, we obtain for the oscillator decay probability, the result

$$|f^{00}(t)|^2 = e^{-\pi g t}\left(1 + \frac{\pi^2 g^2}{4\bar{\omega}^2}\right) + e^{-\pi g t}\frac{8\pi g}{\pi\bar{\omega}^4 t^3}\left(\sin\tilde{\bar{\omega}}t + \frac{\pi g}{2\langle\bar{\omega}\rangle}\cos\tilde{\bar{\omega}}t\right) + \frac{16\pi^2 g^2}{\pi^2\bar{\omega}^8 t^6} \tag{74}$$

where $\tilde{\bar{\omega}} = \sqrt{\bar{\omega}^2 - \frac{\pi^2 g^2}{4}}$. In the above expression the approximation $t \gg \frac{1}{\bar{\omega}}$ plays a role only in the two last terms, due to the difficulties to evaluate exactly the integral in (73) along the imaginary axis. The first term comes from the residue at $\Omega = \tilde{\bar{\omega}} + i\frac{\pi g}{2}$ and would be the same if we had done an exact calculation. If we consider in (74) $g \ll \bar{\omega}$, which corresponds in electromagnetic theory to the fact that the fine-structure constant is small compared to unity, we obtain the well known perturbative exponential decay law for the harmonic oscillator,

$$|f^{00}(t)|^2 \approx e^{-\pi g t}. \tag{75}$$

$N = 2$. Let us call $\Gamma_{11}^{\mu\nu}$ the configuration in which the dressed oscillators q'_μ and q'_ν are at their first excited level and Γ_2^μ the configuration in which q'_μ is at its second excited level. Taking as initial condition that the dressed mechanical oscillator be at the second excited level, the time evolution of the state Γ_2^0 may be obtained in an analogous way to the preceding case:

$$\Gamma_2^0(t) = \sum_\mu \left[f^{\mu\mu}(t)\right]^2 \Gamma_2^\mu + \frac{1}{\sqrt{2}}\sum_{\mu\neq\nu} f^{0\mu}(t) f^{0\nu}(t)\Gamma_{11}^{\mu\nu} \tag{76}$$

where the coefficients $f^{\mu\mu}$ and $f^{0\mu}$ are given by (71). Then it easy to get the following probabilities:

- probability that the dressed oscillator will still be excited at time t:

$$P_0(t) = |f^{00}(t)|^4 \tag{77}$$

- probability that the dressed oscillator has decayed at time t to the first level by emission of a photon:

$$P_1(t) = 2|f^{00}(t)|^2(1 - |f^{00}(t)|^2) \tag{78}$$

- probability that the dressed oscillator has decayed at time t to the ground state:

$$P_2(t) = 1 - 2|f^{00}(t)|^2 + |f^{00}(t)|^4. \tag{79}$$

Replacing (74) in the above expressions we get rigorous expressions for the probability decays. At leading order in g we obtain the well known perturbative formulae for the oscillator decay,

$$P_0(t) \approx e^{-2\pi g t} \tag{80}$$

$$P_1(t) \approx 2e^{-\pi g t}(1 - e^{-\pi g t}) \tag{81}$$

and

$$P_2(t) \approx 1 - 2e^{-\pi g t} + e^{-2\pi g t}. \tag{82}$$

5. Concluding remarks

In this paper we have analysed a simplified version of an atom–electromagnetic-field system and we have tried to give the most exact and rigorous treatment we could to the problem. We have adopted a general physicist's point of view, in the sense that we have renounced an approach very close to the real behaviour of a complicated non-linear system, studying instead a simple linear model. As a counterpart, an exact solution has been possible; our dressed coordinates give a rigorous description of the behaviour of the system . If we expand in powers of the coupling between the mechanical oscillator and the field, we recover the well known behaviour from perturbation theory. We have chosen to take, as the free-space solution to the problem, the limit of the solution inside a spherical cavity as its radius becomes arbitralily large. This choice allows a unified treatment for the system in both the confined and the free-space situations. Moreover, if we start from the confined solution we are able to introduce 'dressed' coordinates to describe exactly the system divided into two parts, the dressed oscillator and field. In the limit of an arbitrarily large cavity (free space) this division is maintained and we have exact formulae to describe the energy flow from the dressed oscillator to the field.

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